ANISOTROPIC DIFFUSION ON COMPLEX TENSOR FIELDS FOR POLSAR IMAGE FILTERING

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ABSTRACT

This paper addresses the problem of denoising for complex tensor image. In particular, we extend the anisotropic diffusion, also known as PM model [1] for filtering images based on PDE, from scalar or vector images to complex tensor ones and apply the new method to remove speckle noises of PolSAR images.

Index Terms— PolSAR images, denoising, anisotropic diffusion

1. INTRODUCTION

Recently, positive definite symmetric matrices has been widely studied and show nice properties in tensor image processing, as covariance matrices for characterizing statistics on deformations [2]. While, the use of Hermitian positive definite matrices (also called complex tensor) has been merely investigated. It is of great interested to exploit the application of Hermitian positive definite matrices for processing polarimetric synthetic aperture radar (PolSAR)images, each pixel of which is usually characterized by complex covariance matrices.

This paper addresses the problem of denoising for complex tensor image. In particular, we extend the anisotropic diffusion(also called PM model) which is a denoising method based on PDE, proposed by Perona and Malik [1] from scalar or vector images to complex tensor images and apply the new method to PolSAR image denoising.

When we extend the anisotropic diffusion to tensor fields, the main challenging problem is that tensor (complex tensor) space is not a vector space and we have to investigate the structure of complex tensor space.

For the purpose of filtering using anisotropic diffusion equation on complex tensor fields, the structure of tensors should be proposed firstly. In [2], Pennec *et. al* proposed an invariant Riemannian metric, exponential and logarithm maps for real tensors. Meanwhile, they deduced an intrinsic geodesic marching scheme for anisotropic filtering based on proposed Riemannian framework for tensor. If we use the anisotropic diffusion equation based on the Riemannian framwork proposed by Pennec *e. al*, there are some problems to be overcome even though it is adequate for complex tensor :

- Computational efficiency When we denoising a complex tensor image by using the anisotropic diffusion equation, we need to compute many SVD for matrices and the time will be consumed very much.
- The gap between definite matrices and semi-definite matrices. The anisotropic diffusion equation is only adequate for definite matrices. For the semi-definite matrix contained zero eigenvalues, logarithm and exponential of the matrix does not exist. This fact causes the PDE cannot be calculated to obtain the denoised tensor image.

Recently, Collard *et. al* [3] proposed an anisotropy preserving metric by using spectral decomposition and quaternion to represent a tensor, which is reported to be more efficient than that of [2]. However, the method in [3] is hard to extend for complex situation. They decomposed the 3-order symmetric definite matrix field $\Sigma(x)$ to $\Sigma = U(x)\Lambda(x)U(x)^T$ by using singular value decomposition firstly and represented the orthogonal matrix $U(x)$ to unit quaternion $\mathbf{q}(x)$ equivalently. Next, they denoise the quaternion $\mathbf{q}(x)$ and the diagonal matrix $\Lambda(x)$ respectively as vector field and diagonal tensor field.

In this paper, we proposed a decomposition on complex tensor fields to overcome the limitation of of the work in [3] on real matrices. We then extend the anisotropic diffusion model from scalar and vector fields to complex tensor fields and obtain the corresponding numerical scheme. Finally, we evaluate our numerical scheme on denoising simulated complex tensor image and real PolSAR image respectively and compare the results to results filtered by using Refined Lee [4].

2. DECOMPOSITION OF COMPLEX TENSORS

In [3], a symmetric positive definite 3×3 matrix is represented as a unit quaternion $\mathbf{q}$ and a diagonal definite matrix Λ. In this section, we extend it to Hermitian positive definite 3×3 matrices.

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Given a Hermitian positive definite 3-order matrix H, we can write it as follow:

$$H = \begin{pmatrix} a_{1,1} & a_{1,2} + b_{1,2}i & a_{1,3} + b_{1,3}i \\ a_{1,2} - b_{1,2}i & a_{2,2} & a_{2,3} + b_{2,3}i \\ a_{1,3} - b_{1,3}i & a_{2,3} - b_{2,3}i & a_{3,3} \end{pmatrix} \quad (1)$$

Using euler's formula, we obtain

$$H = \begin{pmatrix} r_{1,1} & r_{1,2}e^{i\theta_{1,2}} & r_{1,3}e^{i\theta_{1,3}} \\ r_{1,2}e^{-i\theta_{1,2}} & r_{2,2} & r_{2,3}e^{i\theta_{2,3}} \\ r_{1,3}e^{-i\theta_{1,3}} & r_{2,3}e^{-i\theta_{2,3}} & r_{3,3} \end{pmatrix} \quad (2)$$

where $r_{i,j} = \sqrt{a_{i,j}^2 + b_{i,j}^2}$, $\theta_{i,j} = Arg(a_{i,j} + ib_{i,j})$ for $i,j = 1,2,3$.

Theorem 1. *Suppose H is an n-order Hermitian positive definite matrix, the matrix $|H_{i,j}| = (|H_{i,j}|)_{n\times n}$ is a real symmetric matrix.*

Proof. by the definition of Hermitian definite matrix $\forall z \in \mathbb{C}^n$ and $z \neq 0$

$$z^* \Sigma z > 0 \quad (3)$$

Expand the above formula, we have

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^{n} z_i z_j H_{i,j} \\ &= \sum_{i=1}^{n} z_i^2 H_{i,i} + \sum_{i<j} z_i z_j \left(H_{i,j} + H_{i,j}^*\right) \\ &= \sum_{i=1}^{n} z_i^2 H_{i,i} + \sum_{i<j} 2 z_i z_j \Re\left(H_{i,j}\right) \\ &\leq \sum_{i=1}^{n} z_i^2 H_{i,i} + \sum_{i<j} 2 z_i z_j \left|H_{i,j}\right| \\ &= z^T \left|H\right| z \end{aligned} \quad (4)$$

Therefore, the matrix $|H|$ is a real positive definite matrix. □

According to the formula (2) and theorem 1, we will find that 3-order Hermitian matrix H is composed of a real symmetric positive definite matrix and three angles $\theta_{1,2}$, $\theta_{1,3}$ and $\theta_{2,3}$.

Therefore, we can decompose a 3-order Hermitian positive definite matrix H as a symmetric positive definite matrix R and a vector $(\theta_1, \theta_2, \theta_3)^T$. Furthermore, we can represent it as a unit quaternion $\mathbf{q}$, a diagonal matrix Λ and a vector $\Theta = (\theta_1, \theta_2, \theta_3)^T$, written as follows,

$$H \sim (\Lambda, \mathbf{q}, \theta) . \quad (5)$$

3. ANISOTROPIC FILTERING ON COMPLEX TENSOR FIELDS

Anisotropic regularization of images corrupted by noise is very useful in image processing, since it allows for a reduction of the noise level while preserving boundaries and structures. The main idea of anisotropic filtering is to penalize the smoothing in the directions where the derivative is important [1].

In [2], a intrinsic scheme for affine invariant Riemannian metric is deduced. Similarity, we can propose a numerical scheme for the anisotropic processing of complex tensor fields. Our scheme is based on decomposition proposed in Section 2.

For a complex tensor field $H_0(x)$, $x \in \mathbb{R}^2$, we first represent it as $(\Lambda(x), \mathbf{q}(x), \theta(x))$. Then, we use anisotropic filtering on vector fields to $\mathbf{q}(x)$ and $\Theta(x)$. For the diagonal matrices $\Lambda(x)$, we use intrinsic scheme proposed in [2].

The numerical schemes are as follows:

$$\begin{aligned} \Lambda_{n+1}(x) &= \Lambda_n(x) \exp(2\tau \Delta_{aniso} \Lambda(x)) \\ \Delta_{aniso}\Lambda(x) &= \frac{4}{|V(x)|} \sum_{u \in V(x)} c\left(\frac{\|\Lambda_n(x) D_u(x)\|}{\|u\|_2}\right) \\ &\quad \cdot \frac{\Lambda_n(x) D_u(x)}{\|u\|_2^2} \\ D_u(x) &= \log(\Lambda_n(x+u)) - \log(\Lambda_n(x)) \end{aligned} \quad (6)$$

and

$$\begin{aligned} \mathbf{q}_{\text{temp}}(x) &= \mathbf{q}_n(x) + 2\tau \Delta_{\text{aniso}} q(x) \\ \mathbf{q}_{n+1}(x) &= \frac{\mathbf{q}_{\text{temp}}(x)}{\|\mathbf{q}_{\text{temp}}(x)\|} \\ \Delta_{\text{aniso}} q(x) &= \frac{4}{|V(x)|} \sum_{u \in V(x)} c\left(\frac{\|\mathbf{q}_n(x+u) - \mathbf{q}_n(x)\|_2}{\|u\|_2}\right) \\ &\quad \cdot \frac{(\mathbf{q}_n(x+u) - \mathbf{q}_n(x))}{\|u\|_2^2} \end{aligned} \quad (7)$$

and

$$\begin{aligned} \theta_{n+1}(x) &= \theta_n(x) + 2\tau \Delta_{\text{aniso}} \theta \\ \Delta_{\text{aniso}} \theta(x) &= \frac{4}{|V(x)|} \sum_{u \in V(x)} c\left(\frac{\|\theta_n(x+u) - \theta_n(x)\|}{\|u\|_2}\right) \\ &\quad \cdot \frac{\theta_n(x+u) - \theta_n(x)}{\|u\|_2^2} \end{aligned} \quad (8)$$

The function $c(t) = e^{-(\frac{t}{\kappa})^2}$ have different σ in Equation (6),(7),(8).

However, the formula (6) is just adequate for positive definite matrix. In practice, the matrix Λ would be contaminated by noise sometimes. For this problem, we expand the formula

(6), for the $i-th$ diagonal element of $\Lambda_{n+1}(x)$, we have

$$\lambda_{n+1}^{i}(x) = \lambda_n^i(x) \prod_{u\in V(x)} \frac{\exp\left(\frac{8\tau}{|V(x)|} a_u \log \lambda_n^i(x+u)\right)}{\exp\left(\frac{8\tau}{|V(x)|} a_u \log \lambda_n^i(x)\right)}$$
$$= \lambda_n^i(x) \prod_{u\in V(x)} \frac{\exp\left(\frac{8\tau}{|V(x)|} a_u \log \lambda_n^i(x+u)\right)}{\exp\left(\frac{8\tau}{|V(x)|} a_u \log \lambda_n^i(x)\right)}$$
$$= \lambda_n^i(x) \prod_{u\in V(x)} \left(\frac{8\tau}{|V(x)|} a_u\right)^{\lambda_n^i(x+u)} \left(\frac{8\tau}{|V(x)|} a_u\right)^{-\lambda_n^i(x)} \tag{9}$$

where $a_u = c\left(\frac{\|\Lambda_n(x)D_u(x)\|}{\|u\|_2}\right)$, penalizes the smoothing in the direction u at position x. a_u is just adequate for positive definite case, we replace it by

$$\tilde{a}_u = c\left(\frac{\|\Lambda_n(x+u) - \Lambda_n(x)\|}{\|u\|_2}\right), \tag{10}$$

(6) is modified to

$$\lambda_{n+1}^{i}(x)$$
$$= \lambda_n^i(x) \prod_{u\in V(x)} \left(\frac{8\tau}{|V(x)|\,\tilde{a}_u}\right)^{\lambda_n^i(x+u)} \left(\frac{8\tau}{|V(x)|\,\tilde{a}_u}\right)^{-\lambda_n^i(x)}$$
$$i = 1, 2, 3 \tag{11}$$

this scheme is adequate for semi-positive definite case.

Combining the formulas (11),(7) and (8), the algorithm to filter complex tensor images is proposed in algorithm 1.

Algorithm 1: Denoising complex tensor image

Input:
Initial image $\Sigma_0(x, y)$; Iteration numbers N; Iteration step size τ; κ_Λ,$\kappa_{\mathbf{q}}$ and $\kappa_{\arg}$

Output:
Fitered image $\Sigma_n(x)$ after N iterations.

set $k = 0$decompose $\Sigma_0(x, y)$ to $\Lambda_0(x), \mathbf{q}_0(x)\theta_0(\mathrm{x})$
while $k < N$ **do**
  for x **do**
    for $u \in V(x)$ **do**
      compute $\tilde{a}_u = c\left(\frac{\|\Lambda_k(x+u)-\Lambda_k(x)\|}{\|u\|_2}\right)$
      compute $\Delta_{\text{aniso}} q(x) = \frac{4}{|V(x)|} \cdot \sum_{u\in V(x)} c\left(\frac{\|\mathbf{q}_k(x+u)-\mathbf{q}_k(x)\|_2}{\|u\|_2}\right) \cdot \frac{(\mathbf{q}_k(x+u)-\mathbf{q}_k(x))}{\|u\|_2^2}$
      compute $\Delta_{\text{aniso}} \theta(x) = \frac{4}{|V(x)|} \cdot \sum_{u\in V(x)} c\left(\frac{\|\theta_k(x+u)-\theta_k(x)\|}{\|u\|_2}\right) \cdot \frac{\theta_k(x+u)-\arg_k(x)}{\|u\|_2^2}$
    end for
    $\lambda_{n+1}^i(x) = \lambda_n^i(x) \cdot \prod_{u\in V(x)} (\frac{8\tau}{|V(x)|\tilde{a}_u})^{\lambda_n^i(x+u)} (\frac{8\tau}{|V(x)|\tilde{a}_u})^{-\lambda_n^i(x)}$, for $i = 1, 2, 3$
    $\mathbf{q}_{k+1} = \frac{\mathbf{q}_k(x)+2\tau\Delta_{aniso}q(x)}{\|\mathbf{q}_k(x)+2\tau\Delta_{aniso}q(x)\|}$
    $\arg_{n+1}(x) = \arg_n(x) + 2\tau\Delta_{\text{aniso}}\arg$
  end for
  $k := k + 1$
end while
Compose $\Lambda_N(x), \mathbf{q}_N(x)\theta_{\mathrm{N}}(\mathrm{x})$ to $\Sigma_N(x)$
Output $\Sigma_N(x)$

4. EXPERIMENT RESULTS

We evaluates the proposed complex tensor anisotropic filtering method on simulated complex tensor images and real PolSAR image.

4.1. Results on simulated images

For simulated images, we add some speckle noise for ideal images and denoise by using our method and Refined Lee [4] and compute the structural similarity index measurement(SSIM) respectively to compare the results. The results are show in Figure 4.1, Figure 4.2 and Figure 4.3

4.2. Results on real PolSAR images

We choose subparts of San Francisco's PolSAR image and Flevoland's PolSAR image to be denoised and compare our filtering result with Refined Lee [4], the results are showed in Figure 4.

5. CONCLUSION

This paper developed an anisotropic filtering method of PolSAR images using complex tensor, by proposing to represent Hermitian positive definite matrix H as a triple $(\mathbf{q}, D, \Theta)$. We provide a numerical scheme for anisotropic filtering on complex tensor fields, which can perform the anisotropic filtering in an efficient way. Further studies include comprehensive investigations on the properties of the Hermitian positive definite matrix and the proposed anisotropic filtering methods on complex tensor images.

From the experiment results, we find that our method is good at removing noise and keeping edge informations, however, the color information is destroyed somtimes.

6. REFERENCES

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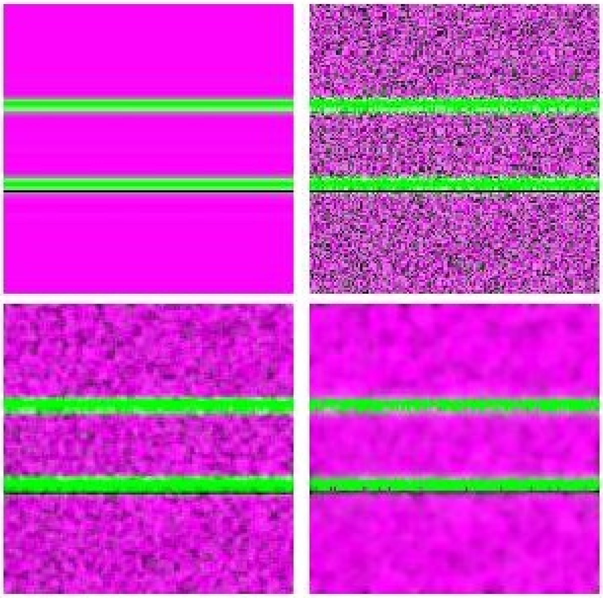

Fig. 4.1. Top-left: clean image, Top-right: add 10% speckle noise; Bottom-left: result by refined Lee with $SSIM = 0.32$, Bottom-rightour result with $SSIM = 0.70$.

Fig. 4.2. Top-left: clean image, Top-right: add 10% speckle noise; Bottom-left: result by refined Lee, $SSIM = 0.7954$, Bottom-rightour result with $SSIM = 0.8071$.

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Fig. 4.3. Left Top : Ideal image, Right Top: add 10% speckle noise; Left Bottom : Refined Lee, SSIM = 0.7625 , Right BottomOur result, SSIM = 0.7959

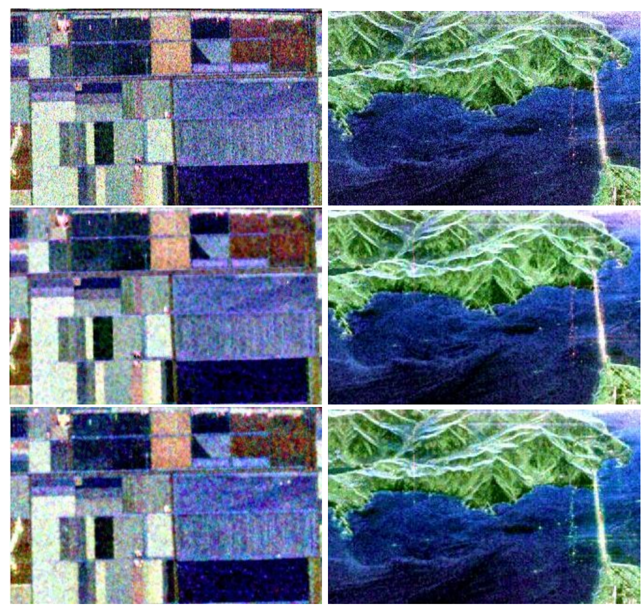

Fig. 4. Comparison of the results on a subpart of Flevoland's PolSAR image (top-left) and a subpart of San Francisco's PolSAR image (top-right). Mid-line: the resuls by refined Lee. Bottom-line: results by our method.

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